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Note

Planarity of iterated line graphs

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Abstract

The *line index* of a graph G is the smallest k such that the kth iterated line graph of G is nonplanar. We show that the line index of a graph is either infinite or it is at most 4. Moreover, we give a full characterization of all graphs with respect to their line index.
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1. Introduction

We consider the problem of planarity of line graphs. The main motivation for this work is a result of Sedláček [3] which characterizes graphs whose line graph is planar:

Theorem 1 (*Sedláček [3]*). *The line graph of a graph G is planar if and only if G is planar, $\Delta(G) \leqslant 4$, and every vertex of degree 4 in G is a cut-vertex.*

Using Sedláček's characterization, Greenwell and Hemminger [2] give a forbidden subgraph characterization of graphs with planar line graph. In the following theorem, $G \vee H$ denotes the join of graphs G and H, and $\overline{G}$ denotes the complement of G.

Theorem 2 (*Greenwell and Hemminger [2]*). *The line graph of a graph G is planar if and only if G has no subgraph homeomorphic to $K_{3,3}$, $K_{1,5}$, $P_4 \vee K_1$ or $K_2 \vee \overline{K}_3$.*

Deogun [1] characterizes regular planar line graphs.

We study a generalization of the problem, namely the planarity of the iterated line graphs of a graph. Given a graph G, we denote the kth iterated line graph of G by $L^k(G)$. In particular $L^0(G) = G$ and $L^1(G) = L(G)$ is the line graph of G.

Definition 3. The *line index* of a graph G is the smallest k such that $L^k(G)$ is nonplanar. We denote the line index of G by $\xi(G)$. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$.

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It is shown in [3] that if G is nonplanar then $L(G)$ is also nonplanar. That implies:

$\xi(G) = 1 + \max\{k \mid L^k(G) \text{ is planar}\}$.

We show that the line index of every graph is either ∞ or it is at most 4. Moreover, our main result (Theorem 10) completely characterizes all graphs with respect to their line index.

2. Structural characterization

The function ξ is a non-increasing function with respect to the subgraph order. This fact is easy to prove, however, since it is used several times throughout this paper, we state it formally:

Lemma 4. *If H is a subgraph of G then $\xi(G) \leqslant \xi(H)$.*

Proof. Let $k = \xi(H)$. Then $L^k(H)$ is nonplanar and therefore $L^k(G)$ is nonplanar since $H \leqslant G$. This implies $\xi(G) \leqslant k$. $\square$

The following is an immediate corollary of the above lemma.

Lemma 5. *If G is a graph with $\Delta(G) \geqslant 4$ then $\xi(G) \leqslant 3$.*

Proof. If $\Delta(G) \geqslant 4$ then G has $K_{1,4}$ as a subgraph. On the other hand $L^3(K_{1,4})$ is a 6-regular graph and since every simple planar graph has minimum degree at most 5, $L^3(K_{1,4})$ is nonplanar. Therefore $\xi(K_{1,4}) \leqslant 3$ and by Lemma 4 we have $\xi(G) \leqslant 3$. $\square$

Indeed, $L^2(K_{1,4})$ is isomorphic to the octahedron graph which is planar and therefore $\xi(K_{1,4}) = 3$. By Lemma 4 the line index of a graph is the maximum of the line indices of its connected components. Hence we may assume that from here on all graphs are connected.

Theorem 6. *For any graph G, we have $\xi(G) \in \{0, 1, 2, 3, 4, \infty\}$. Moreover, $\xi(G) = \infty$ if and only if G is a path, a cycle, or $K_{1,3}$.*

Proof. Let P_n and C_n denote, respectively, the path and cycle on n vertices and let P_0 be the empty graph, namely a graph with no vertices and no edges. We know $L(P_n) = P_{n-1}$ for $n \geqslant 1$, $L(P_0) = P_0$, $L(C_n) = C_n$, and $L(K_{1,3}) = C_3$. Therefore if G is a path, a cycle, or $K_{1,3}$, then $\xi(G) = \infty$. Next, assume that G is neither a path, a cycle, nor $K_{1,3}$. Since G is connected and it is not a path or a cycle, $\Delta(G) \geqslant 3$. If $\Delta(G) \geqslant 4$, by Lemma 5 we have $\xi(G) \leqslant 3$. Let Y_2 be the graph obtained by subdividing one of the edges of $K_{1,3}$. If $\Delta(G) = 3$, since G is not $K_{1,3}$, it has Y_2 as a subgraph. It may be easily verified that $\xi(Y_2) = 4$ and hence $\xi(G) \leqslant 4$ by Lemma 4. $\square$

A graph has line index 0 if and only if it is nonplanar. Theorems 1 and 2 give two different characterizations of graphs with line index 1. Graphs with line index ∞ are characterized in Theorem 6. In the rest of the paper we give characterizations of graphs with line index 2 and 4. This also characterizes graphs with line index 3 since that is the only remaining case.

Theorem 7. *Let G be a graph. Then $\xi(G) = 2$ if and only if $L(G)$ is planar and G has a subgraph isomorphic to one of the graphs H_i in Fig. 1.*

Proof. Let G be a graph with $\xi(G) = 2$. Then by the definition, $L(G)$ is planar while $L^2(G)$ is nonplanar. By Theorem 1, $L(G)$ either has a vertex e with degree at least 5 in which case G contains H_1 as a subgraph, or it has a vertex e with degree 4 which is not a cut-vertex. In the latter case, the vertex e of $L(G)$ is an edge of G which is not a bridge. Hence G has a cycle containing e. Since e has degree 4 in $L(G)$, G contains either H_2, H_3 or H_4 as a subgraph. To prove the converse, let $H \leqslant G$ be one of the graphs in Fig. 1. If $H = H_1$ then $L(H)$ has a vertex of degree at least 5, and if $H \in \{H_2, H_3, H_4\}$ then $L(H)$ has a vertex of degree 4 which is not a cut-vertex. Therefore $L^2(H)$ is nonplanar by Theorem 1, which implies $L^2(G)$ is nonplanar. Since $L(G)$ is planar it follows that $\xi(G) = 2$. $\square$

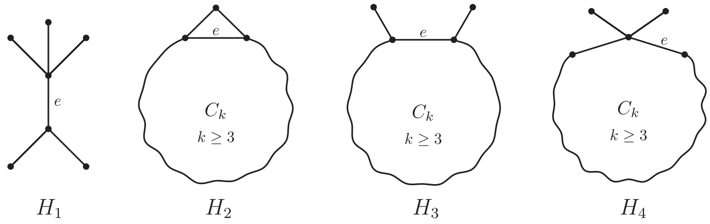

Fig. 1. Forbidden subgraphs for $\xi \geqslant 3$.

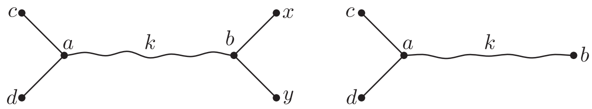

Fig. 2. The graphs X_k and Y_k.

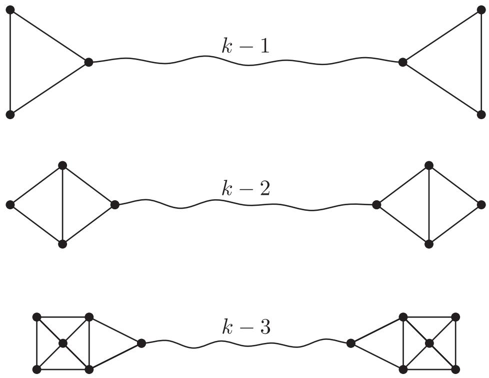

Fig. 3. The graphs $L(X_k)$ and $L^2(X_k)$ for $k \geqslant 2$, and $L^3(X_k)$ for $k \geqslant 3$.

Remark 8. For every $i = 2, 3, 4$ and $k \geqslant 3$, let $H_i(k)$ be the graph H_i of Fig. 1 with parameter k. Note that if $k \geqslant 4$, the graph $H_2(k)$ has a subgraph isomorphic to $H_3(3)$. Thus a minimal set of forbidden subgraphs for $\xi \geqslant 3$ can be obtained by replacing H_2 in Fig. 1 by the graph $H_2(3)$.

Let G be a complete bipartite graph with the partite sets $\{a\}$ and $\{b, c, d\}$. Let k be a positive integer. We define Y_k to be the graph obtained from G by subdividing the edge ab to a path of length k. We define X_k to be the graph obtained by adding two new vertices x and y to Y_k and joining them to b. These two families of graphs are depicted in Fig. 2. Similarly we define T_k to be the graph obtained from G by subdividing the edges ab and ac to paths of length k.

Note that Y_1 and T_1 are isomorphic to $K_{1,3}$ and hence they both have line index ∞. On the other hand one may easily verify that the graphs X_1 and T_k for $k \geqslant 2$ all have line index 3. In fact by Theorem 10, the only subdivisions of $K_{1,3}$ with $\xi \neq 3$ are the graphs Y_k with $k \geqslant 1$.

Theorem 9. *For a graph G, $\xi(G) = 4$ if and only if G is one of the graphs X_k or Y_k with $k \geqslant 2$.*

Proof. As shown in Figs. 3 and 4, if $k \geqslant 2$ then $L^i(X_k)$ is planar for $i \in \{1, 2, 3\}$. Thus, by Theorem 6, $\xi(X_k) = 4$. Also, since Y_k is a subgraph of X_k, Lemma 4 implies that $\xi(Y_k) = 4$.

For the converse, let G be a graph for which $\xi(G) = 4$. By Lemma 5 and Theorem 6, $\varDelta(G) = 3$. Since G is connected and not a cycle, if G has a cycle C, then at least one vertex of C has degree $\geqslant 3$ in G. Hence either G or $L(G)$ contains one of the configurations H_2 and H_3 of Fig. 1 and by Theorem 7 we have $\xi(G) \leqslant 3$. This contradiction implies G is

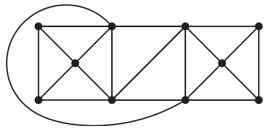

Fig. 4. The graph $L^3(X_2)$.

a tree. On the other hand since $\xi(T_2) = 3$, G does not have T_2 as a subgraph. It is easy to see that the only trees with maximum degree 3 and no subgraph isomorphic to T_2 are the graphs X_k and Y_k. Finally we must have $k \geqslant 2$ since $\xi(X_1) = 3$ and $\xi(Y_1) = \infty$. $\square$

The following theorem is a summary of the results of this section, giving a full characterization of graphs with respect to their line index.

Theorem 10. *Let G be a connected graph. Then*:

- $\xi(G) = 0$ *if and only if G is nonplanar.*
- $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or* $K_{1,3}$.
- $\xi(G) = 1$ *if and only if G is planar and either* $\Delta(G) \geqslant 5$ *or G has a vertex of degree* 4 *which is not a cut-vertex.*
- $\xi(G) = 2$ *if and only if* $L(G)$ *is planar and G contains one of the graphs* H_i *in Fig.* 1 *as a subgraph.*
- $\xi(G) = 4$ *if and only if G is one of the graphs* X_k *or* Y_k *for some* $k \geqslant 2$.
- $\xi(G) = 3$ *otherwise.*

Note that by Theorem 2, a forbidden subgraph classification of graphs with line index 1 follows: $\xi(G) = 1$ if and only if G is planar and G has a subgraph homeomorphic to $K_{1,5}$, $P_4 \vee K_1$ or $K_2 \vee \overline{K}_3$.

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